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An approach to orbital mechanics, which is accessible to beginning physics students and presupposes no knowledge of calculus, is presented. A theory of orbits is developed for the inverse-square central force law which differs considerably from the usual deductive approach. This document begins with qualitative aspects of solutions, and leads to a number of geometrically realizable physical invariants of the orbits. Consequently, most of the theorems rely only on simple geometrical relationships. Despite its simplicity, this planetary geometry is powerful enough to treat a wide range of perturbations with relative ease. It is felt that this treatment provides a better view of "what doing physics is really like" than the standard route via algebraic manipulations. The document concludes with suggestions for further research into the geometry of planetary orbits. (MP)

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VELOCITY SPACE AND THE GEOMETRY OF PLANETARY ORBITS*

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ABSTRACT

We develop a theory of orbits for the inverse-square central force law which differs considerably from the usual deductive approach. In particular, we make no explicit use of calculus. By beginning with qualitative aspects of solutions, we are led to a number of geometrically realizable physical invariants of the orbits. Consequently most of our theorems rely only on simple geometrical relationships. Despite its simplicity, our planetary geometry is powerful enough to treat a wide range of perturbations with relative ease. Furthermore, without introducing any more machinery, we obtain full quantitative results. The paper concludes with suggestions for further research into the geometry of planetary orbits.

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I. Some Qualitative Results

1. *Introduction*

From junior high school on, students of science are taught that Kepler's Laws describe the motion of planets around the sun. They are given no hint of how they themselves can understand the "why" of these laws. By high school the students have been taught Newton's discovery, that the inverse-square force law accounts for those beautiful ellipses, but the connection is not yet for their eyes. After a year or so of college it's finally time to plow through the thoroughly standardized and unmotivated proofs, using intricate manipulations with differential equations.

In this paper we outline an approach to orbital mechanics which is accessible to beginning physics students and presupposes no knowledge of calculus. We give an elementary (yet mathematically correct) treatment of Kepler's Laws and also investigate a simple first-order perturbation theory for orbits in an inverse-square field. Our theorems and proofs arise naturally from trying to understand orbits in terms of their physical invariants. We therefore feel that our treatment provides a better view of "what doing physics is really like" than does the standard route via algebraic manipulations.

The key to the method lies in consdering the velocity space picture for a planet's motion about the sun. The concept of a velocity space is not normally encountered by the

student until he or she is presented with phase space in the context of the formalities of Hamiltonian mechanics. We think there is little reason for this delay because, though it is usually considered an "advanced concept", the velocity space picture of a particle's motion lies at the core of Newtonian mechanics. Indeed, the qualitative content of Newton's F=ma (for our purposes, $m\Delta v=F\Delta t$) is simply that physical interactions between objects take place by a modification of velocity, rather than by a change in position. Appreciation of this fact can greatly assist the development of physical intuition and understanding, and velocity space is a natural tool for exploring Newton's primarily conceptual breakthrough. In Part III, when we discuss perturbations, we shall see what rich dividends can be yielded by looking at physical phenomena in the right conceptual frame--in this case, velocity space.

In the following presentation, we have tried to walk rather a narrow path between two extremes. On one hand, a description of our methods and results would take no more than a few pages if we used the full precision of mathematical apparatus (including calculus) available to science students after a few years of university education. On the other hand, we could have spent considerably more space developing a complete and self-contained course for very early physics students. Since we feel the material can be useful at both levels of physics education, we have attempted a compromise. We apologize both to those who find our presentation extended and perhaps verbose, and to those who might find it sketchy and incomplete.

We gratefully acknowledge the inspiration and encouragement of Seymour Papert. He introduced us to this way of thinking about orbits, and pointed out the basic results described in Sections 2 and 3. These sections closely follow parts of his paper (reference 1) which presents a broader view of the conception of education in science and mathematics from which this work grew. We would like to thank the editor and referee from *The American Journal of Physics* for many encouraging and helpful comments and also to thank Suzin Jabari of the M.I.T. Artificial Intelligence Laboratory for preparing the illustrations for this paper.

2. *The Orbit is Closed*

Standard approaches begin with the arduous task of proving that planetary orbits are precise ellipses. We begin by proving a more qualitative proposition, that orbits are closed. In doing so we dispense with a great deal of analytic clutter, and the important special nature of inverse-square orbits which makes them closed comes into central focus.

We will prove that no orbit like that in Figure 1 is possible.

> *If a planet crosses a half-line from the sun twice, then it crosses it at the same point each time—not further out or closer in.*

We assume two pieces of knowledge:

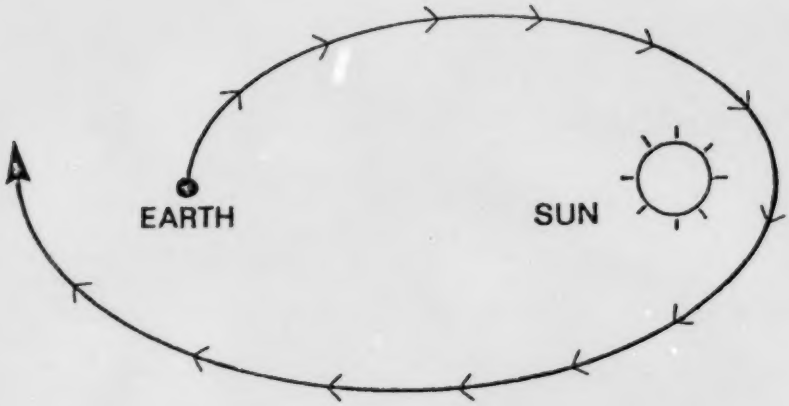


Figure 1: An impossible orbit.

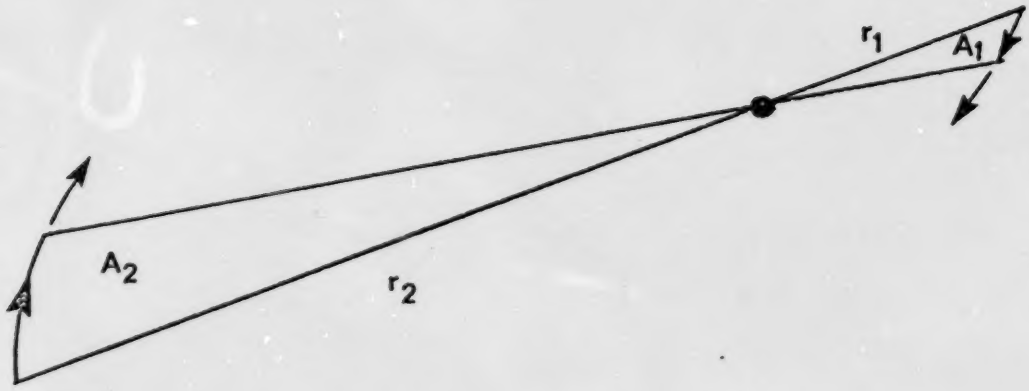


Figure 2: Opposite pieces of orbit, at radii r_1 and r_2, sweep out areas A_1 and A_2 in times Δt_1 and Δt_2.

1. The force on the planet, when it is distance r from the sun, is K/r^2 towards the sun.

2. Angular momentum is conserved. We use this in the form of Kepler's Law that the radius from the sun to a planet sweeps out equal areas in equal times. This can be easily derived and we remind readers of its simple geometric proof in the appendix.

Now consider diametrically opposite pieces of the orbit which subtend the same (small) angle measured from the sun, as in Figure 2. Kepler's law states that

$$\frac{A_1}{A_2} = \frac{\Delta t_1}{\Delta t_2}$$

What else do we know about the area or the time? Geometry tells us that

$$\frac{A_1}{A_2} \qquad \frac{r_1^2}{r_2^2}$$

Those r^2's are too suggestive for us not to make a connection with $F = K/r^2$. In fact

$$\frac{F_2}{F_1} = \frac{r_1^2}{r_2^2} \text{ hence } \frac{F_2}{F_1} = \frac{A_1}{A_2} = \frac{\Delta t_1}{\Delta t_2},$$

and we conclude $F_1 \Delta t_1 = F_2 \Delta t_2$. Since $\vec{F}_1$ and $\vec{F}_2$ pull in opposite directions,

$$\vec{F}_1 \, \Delta t_1 = -\vec{F}_2 \, \Delta t_2$$

We can identify these terms: According to Newton's Second Law, on each piece of the orbit, $\vec{F}\Delta t$ is precisely $\Delta\vec{v}$ the change in velocity[2] which we call the "kick" associated to that piece. Thus the last equation says that the change in velocity over one piece of orbit exactly cancels the change on the opposite piece. Starting on the half-line which the orbit crosses twice, divide the orbit all the way around into similar pairs of opposite pieces. The total change in the planet's velocity between successive crossings of the half-line is the sum of the changes in velocity over each small piece; adding these up in opposing pairs, we see that the total change is zero. Whenever the planet crosses a given half-line, it has the same velocity.

Now Kepler's dictum of equal area in equal time allows us to conclude that at two crossings of the half-line, not only velocity but distance from the sun is the same. Figure 3 shows the areas swept out by the planet in some short time Δt after successive crossings of the half-line through A, B, and O. The velocities at A and B are equal. Therefore the pieces of orbit AC and BD are both equal to $\vec{v}\Delta t$, but the area of AOC must equal the area of BOD. Then A equals B and the orbit closes.

3. *A Theorem in Velocity Space*

The preceding proof rested on the fact that the kicks over opposite pieces of the orbit have equal magnitudes:

$$F_1 \Delta t_1 = F_2 \Delta t_2$$

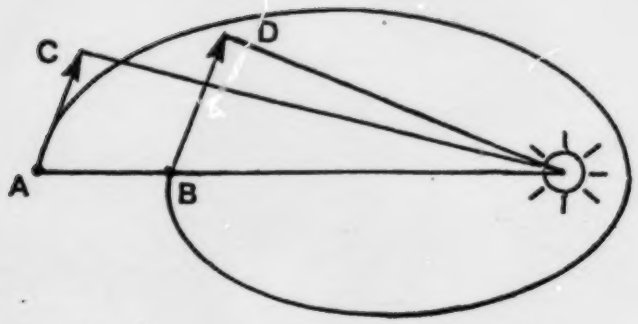


Figure 3: AC=BD, area AOC=area BOD, therefore OA=OB and A=B.

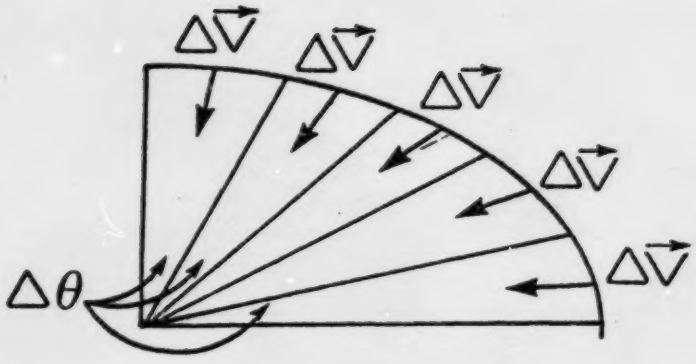


Figure 4: Part of an orbit divided into equal-angle slices.

But, to derive this equation it isn't necessary that the pieces of orbit be opposite. We need only that

$$\frac{A_1}{A_2} = \frac{r_1^2}{r_2^2} ,$$

and this is true for any two pieces of the orbit over which the radial angle changes by the same small amount. So, if we divide the whole orbit into small pieces subtending the same angle $\Delta\theta$, the "kick" vectors for the various pieces all have the same length (Figure 4). Not only are all the lengths equal, but the rotation between sucessive kick vectors is constant and equals $\Delta\theta$.

Thus we have a very simple algorithm for generating the changing velocity as the planet moves along its orbit. Starting at a given velocity vector we add on kick vectors one after another. Each addition is a step of constant length and successive steps differ in direction by the constant turn, $\Delta\theta$. It is easy to see that the algorithm: GO FORWARD a short distance, TURN through a small angle, GO FORWARD the same short distance, TURN through the same small angle,..., will generate a circle.[3] We conclude that the kick vectors line up along a circle (Figure 5).

We can interpret "adding on successive kick vectors" by introducing the notion of *velocity space*. The velocity of an object is usually described by a "velocity vector", that is, a

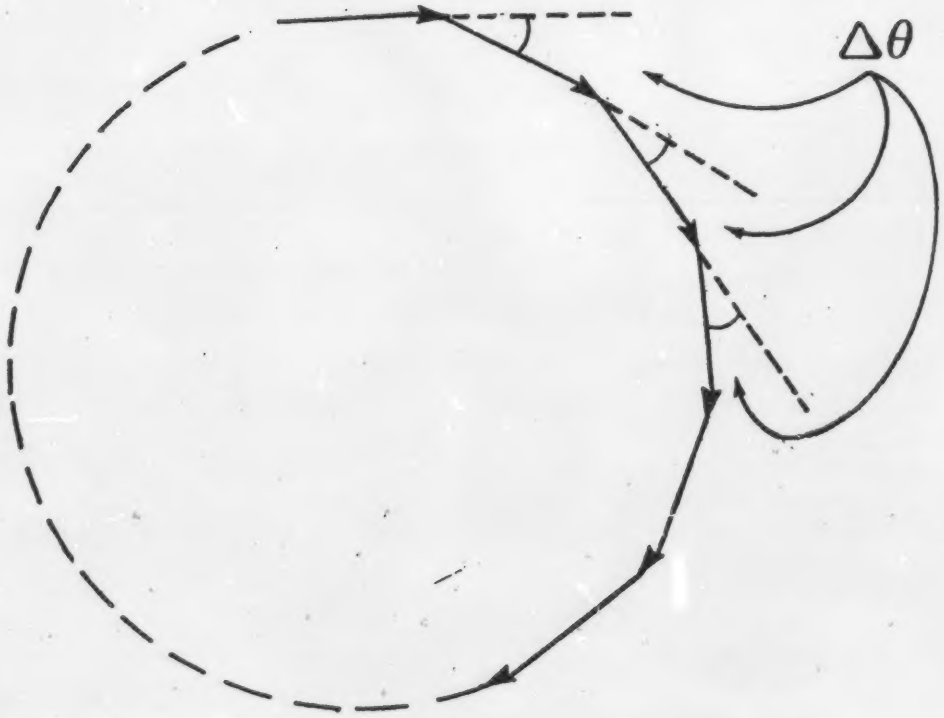


Figure 5: Placing the kick vectors end to end.

direction and a speed (length). In comparing different velocities it is useful to put the tail of all velocity vectors down at some common point, 0, and to depict a velocity by the point where the tip of the velocity vector lands. With this convention, we can draw two different pictures to describe the motion of an object:

(1) the collection of successive positions of the object in "real" space; and

(2) the collection of successive positions of the tip of the object's velocity vector. This is a path in "velocity space", a picture of how velocity changes.

The second picture is called the "velocity space path" or "velocity diagram". Figure 6 exemplifies these two kinds of diagrams.

Velocity is the thing that changes position; *kicks* are the things that change velocity. To get from an object's position at one instant, t, to its position at $t+\Delta t$, we add on the vector $\vec{v}\Delta t$. To get from the object's velocity at one instant to the velocity at a slightly later time, we add on a kick vector, $\vec{F}\Delta t$. Adding up successive $\vec{v}\Delta t$ vectors gives the position space path; adding kick vectors gives the velocity space path. We can now restate our result as

Circle Theorem

For an object moving in an inverse-square field, the velocity space path lies on a circle.

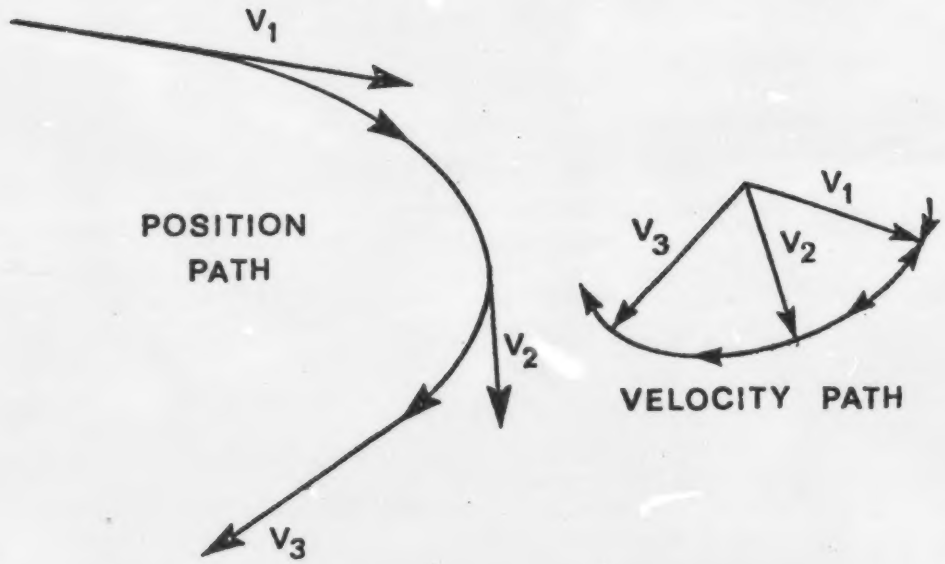


Figure 6: One object's position space and velocity space paths.

To avoid confusion we point out that the center of this circle is *not* necessarily at the origin in velocity space.

4. *The Velocity Space Path*

Our Circle Theorem tells us that the velocity space path lies on a circle. But is it a complete circle or just part of it? We can answer that question and a bit more.

We showed that an orbit which does manage to get all the way around the sun crosses every half-line from the sun exactly once. Such an orbit is a simple closed curve. In a complete revolution, therefore, the direction of the planet's velocity vector must change through 360°. (See reference 4.) That means that *in velocity space also* the path meets every half-line from the velocity space origin. It follows that closed orbits in position space correspond to complete circles in velocity space, and we have learned, besides, that the origin of velocity space is inside the circle.

We now have a good qualitative picture of the velocity space path for a closed orbit. (Open orbits are discussed in section II.) In Part II, we will extract information about the position space orbit from our velocity space diagram.

II. Invariants of the Orbit

5. *Angular Momentum*

As we have seen, the velocity space path of a planet in an inverse-square field lies on a circle. One obvious invariant of a circle is its radius. How can we interpret this invariant physically?

We got the circle in section 3 as the result of the algorithm "forward distance D, turn angle $\Delta\theta$, repeat." As one can see (Figure 7), this generates a circle of radius $D/\Delta\theta$. In our case $\Delta\theta$ was an arbitrary small angle and D was the magnitude of the kick $F\Delta t$ over the corresponding small piece of orbit. Letting u denote the radius of the velocity circle, we have

$$u = \frac{F\Delta t}{\Delta\theta}$$

It is not immediately obvious that this is a constant. However, we can simplify the expression using the fact from geometry that the area swept out over a small piece of orbit is

$$A = \frac{1}{2} r^2 \Delta\theta.$$

Then $u = \frac{F\Delta t}{\Delta\theta} = \frac{Fr^2\Delta t}{2A}$. We can eliminate the apparent dependence on the non-constant term r^2 by using $F = K/r^2$; we obtain

$$u = \frac{K}{\left(2A/\Delta t\right)}$$

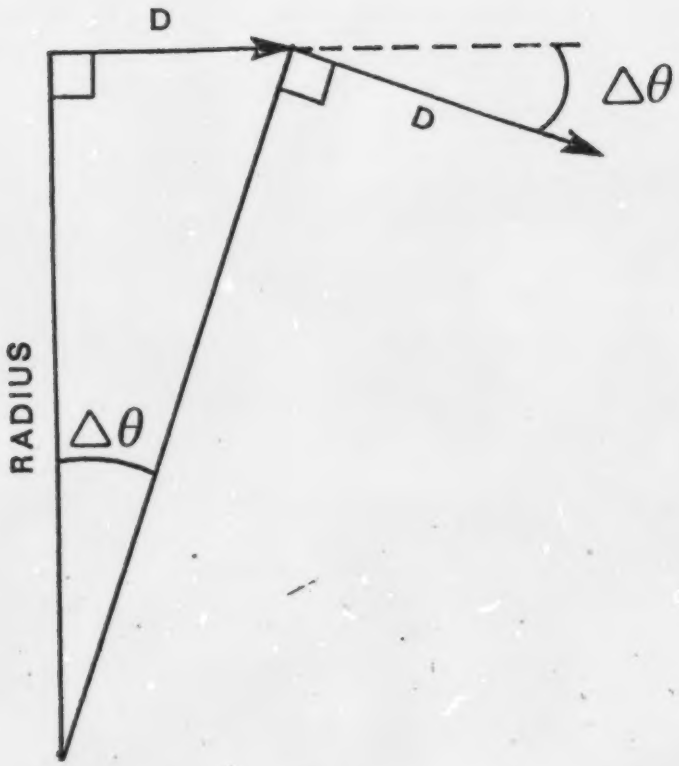


Figure 7: D = radius x $\Delta\theta$

The term $2A/\Delta t$, which tells how fast the planet is sweeping out area, is precisely the constant called *angular momentum*, L. (See Appendix.) Therefore

The radius of the velocity circle equals the force constant

K divided by the angular momentum L: u=K/L

For a fixed gravitational field, the radius of the velocity circle tells us the planet's angular momentum: a larger radius gives a smaller angular momentum.

6. *Orientation*

The velocity circle has another invariant so obvious it is easy to overlook--the position of its center in velocity space. As we remarked above, even though the origin 0 is *inside* the velocity circle it need not be *at the center* of the circle. Let $\vec{z}$ be the vector running from the origin in velocity space to the center of the velocity circle, and let $\vec{u}$ be a radial vector of the circle (Figure 8). In terms of $\vec{z}$ and $\vec{u}$, we can think of the planet's path in velocity space as follows: at each moment the velocity $\vec{v}$ is the sum of a *constant vector* $\vec{z}$ and a vector $\vec{u}$ of *constant length* (equal to u=K/L), $\vec{v}=\vec{z}+\vec{u}$. The velocity space path is generated as the radius $\vec{u}$ sweeps around the tip of the invariant vector $\vec{z}$

There is a quite remarkable relation between the motion of $\vec{r}$, the position space radius vector (tail at the sun, head at the planet), and the motion of this "velocity space radius," $\vec{u}$.

Correlation of Angles in Position and Velocity Space

At each moment the planet's radial vector $\vec{r}$ is perpendicular to the

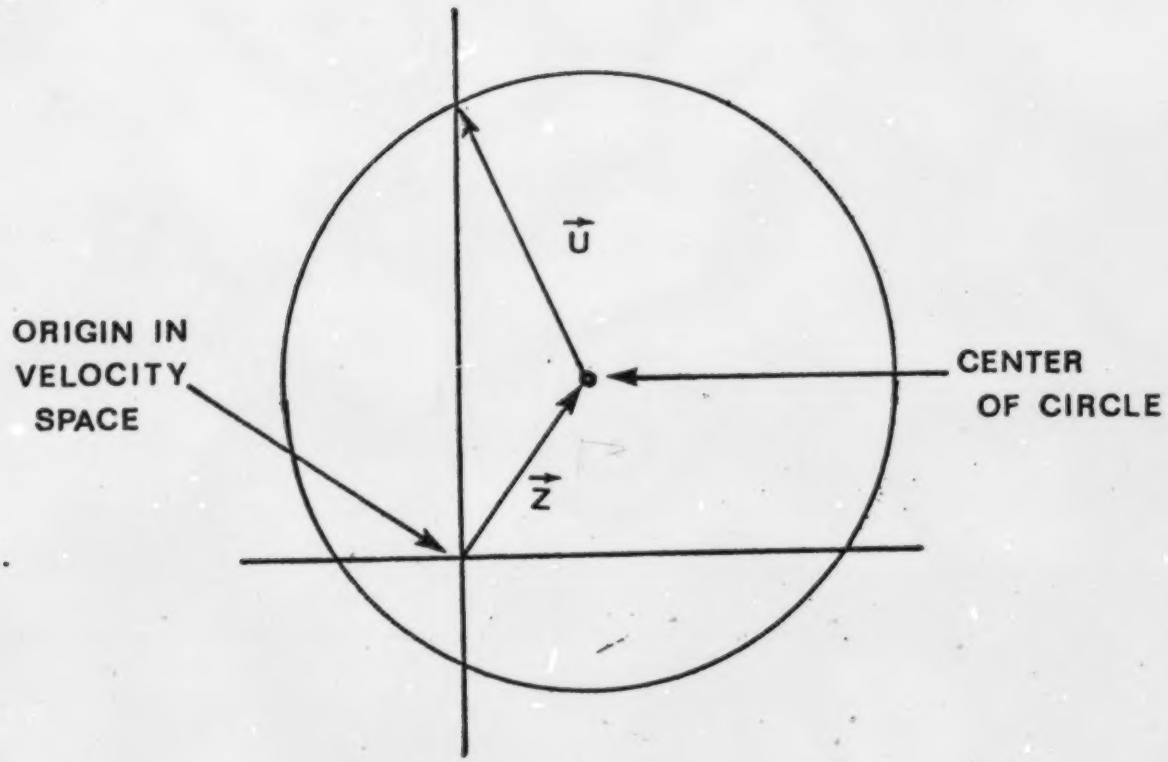


Figure 8: The invariant $\vec{z}$.

radius $\vec{u}$ of the velocity circle.

To see this we examine how the kicks fit into both diagrams: In position space each kick is parallel to the radial vector $\vec{r}$. In velocity space the same kick is tangent to the velocity circle and hence perpendicular to the velocity circle radius $\vec{u}$.

Hence $\vec{u}$ is perpendicular to $\vec{r}$.

Correlation of Angles is a powerful principle. It tells us:

(1) Each point on the velocity space path corresponds to a unique point in the planet's orbit. (The planet cannot attain the same velocity at two different points in its orbit.)

(2) The planet's position vector sweeps around the sun at the same rate and with the same direction (clockwise or counter- clockwise) as its velocity sweeps around the circle in velocity space. The two are always 90 degrees out of phase. (Figure 9)

Now we can give more meaning to the $\vec{z}$ vector. The planet's speed, v, is greatest when the $\vec{u}$ and $\vec{z}$ vectors are lined up, least when they are opposed: $v_{min}=u-z$, $v_{max}=u+z$. Therefore the $\vec{z}$ vector points in the direction of maximum speed and opposite to the direction of minimum speed.

The points of greatest and least speed occur where the velocity vector is parallel to $\vec{u}$, and so perpendicular to the position space radius $\vec{r}$. It is not hard to show then that the point where speed attains its maximum (respectively, its minimum) corresponds to the minimum (respectively, maximum) distance from the sun. The reader can fill in the details of the proof sketched below:

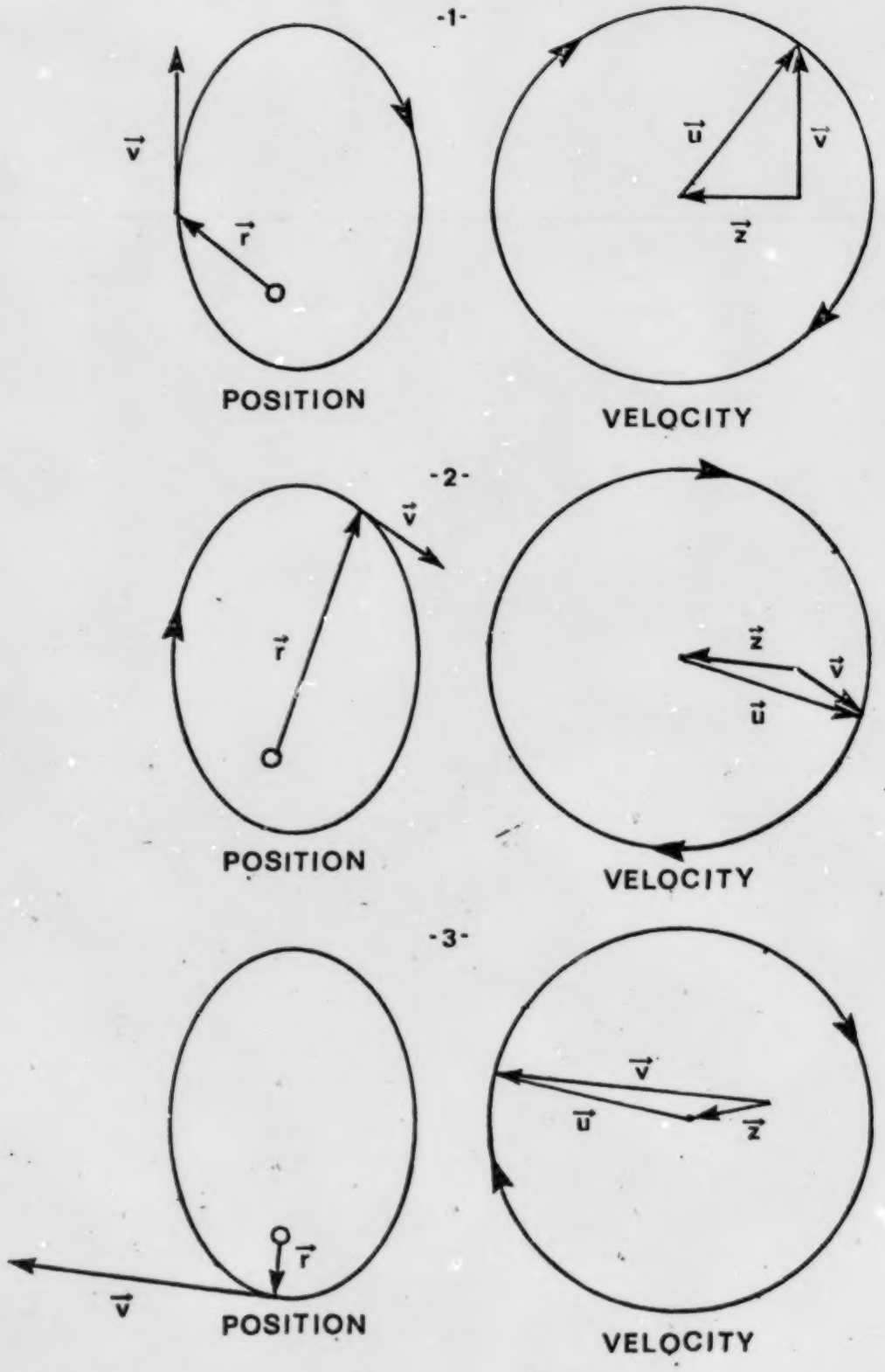


Figure 9: Snapshots of position and velocity over an orbit $\vec{u} \perp \vec{r}$.

Step 1: At a maximum or minimum distance the velocity $\vec{v}$ must be perpendicular to the radius $\vec{r}$.

Step 2. The velocity diagram shows that there are precisely two points where this can occur.

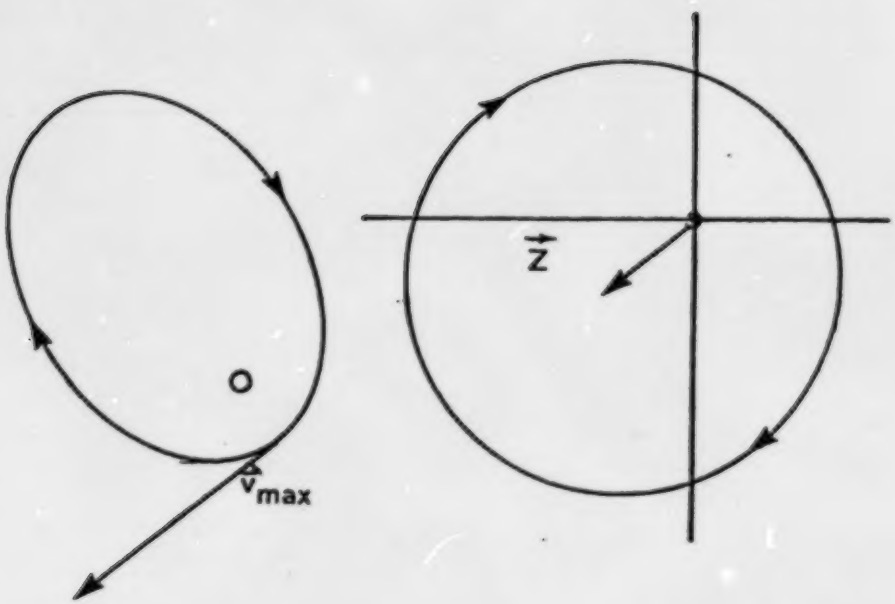
Step 3. Conservation of Angular Momentum implies that r_{max} corresponds to v_{min} and r_{min} to v_{max}.

Now we have some more qualitative information about the shape of the orbit: there is precisely one point of maximum distance from the sun, and one of minimum distance. They occur on opposite sides of the sun since the corresponding $\vec{u}$ vectors point in opposite directions. The $\vec{z}$ vector determines the orientation of the orbit. It points in the direction of maximum speed. (Figure 10)

7. *Shape and Symmetry*

What more does the length of $\vec{z}$ tell us about the orbit? Consider what would happen if $\vec{z}$ vanished. There would then be no direction picked out for maximum speed or distance from the sun. The planet would have to travel around the sun in a circle at uniform speed. (Another way to see this: $\vec{v}$ would be equal to $\vec{u}$ and therefore always perpendicular to $\vec{r}$ and of constant length--the characteristic of uniform circular motion.)

This suggests that $\vec{z}$ indicates how the orbit deviates from a circle. We can make this precise. One obvious measure of the non-circularity of the orbit is the difference in the extreme distances from the sun,



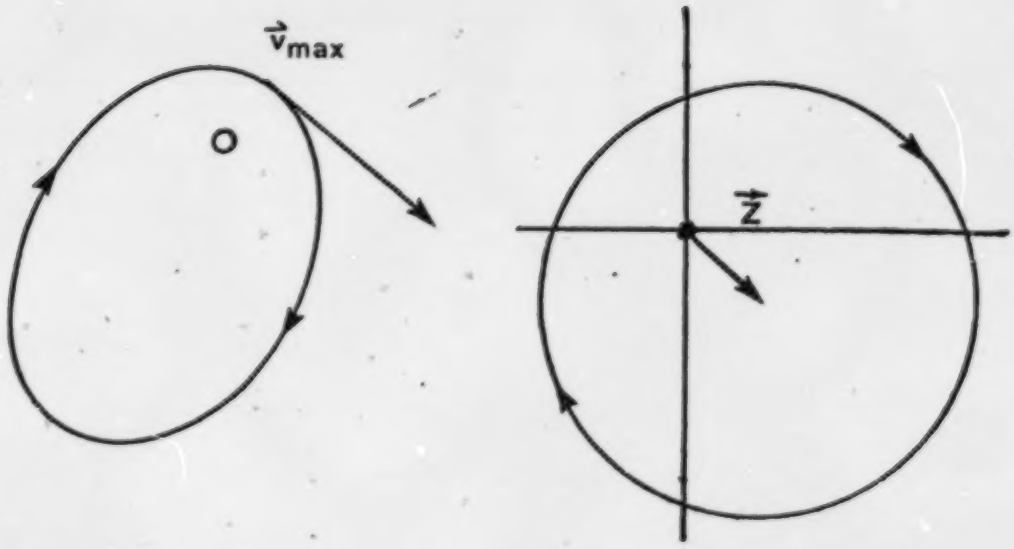


Figure 10: $\vec{z}$ determines the orientation of the orbit.

$r_{max}-r_{min}$.

If we want an invariant that depends only on the shape and not the size of the orbit it is better to see how much the ratio

$$r_{max}/r_{min}$$

differs from 1. The length of $\vec{z}$, relates the maximum and minimum speeds:

$$v_{min}=u-z \quad v_{max}=u+z$$

To relate speed to distance from the sun we use angular momentum. If the angular momentum is L then

$$L=v_{min} \times r_{max}=v_{max} \times r_{min}$$

because at these places in the orbit v is perpendicular to r (Section 6). Therefore

$$\frac{r_{max}}{r_{min}} = \frac{V_{max}}{V_{min}} = \frac{u+z}{u-z}$$

Since u=K/L we can also write

$$\frac{r_{max}}{r_{min}} = \frac{K+Lz}{K-Lz}$$

But K depends only on the nature of the gravitational field so we see that our "shape invariant" is determined by Lz. The larger Lz, the more the orbit deviates from a circle. In section II we will derive the analytic result that the orbit is an *ellipse* and Lz determines its *eccentricity*.

In fact, knowledge of K and the velocity diagram essentially determines the planet's motion. The maximum velocity can be read off the diagram immediately, as can u=K/L; so we know L. The shortest radius in position space has length $r_{min}=L/v_{max}$, and

is perpendicular to $\vec{v}_{max}$ (Section 6). Having this one vector, $\vec{r}_{min}$, we can generate the orbit starting at the position determined by $\vec{r}_{min}$ with the following algorithm.

1. Travel a short distance $\vec{v}\Delta t$ in the direction of $\vec{v}$.

2. Measure the change in angle, $\Delta\theta$, in position space.

3. Find the velocity at this new angle (by rotating $\vec{u}$ through $\Delta\theta$ and consulting the velocity diagram).

4. Return to step 1. This generates the entire position space path.

Notice that the velocity diagram is symmetric about the line determined by $\vec{z}$. The above algorithm translates this fact into a symmetry of the position space orbit. Starting at the nearest point to the sun, construct the orbit in the forward direction for a while, along $\vec{v}_1$ for $\Delta\theta_1$, then along $\vec{v}_2$ for $\Delta\theta_2$, and so on. Now go back to the starting point and run the algorithm backwards with the same sequence of $\Delta\theta$'s. Since the velocity diagram is symmetric we generate the same small segments of orbit, except that they have been flipped about the line perpendicular to $\vec{z}$. Therefore the entire orbit is symmetric about this line.

8. *Summary*

We have so far obtained the following information from the velocity diagram.

1. The radius of the velocity circle determines the orbit's angular momentum: u=K/L.

2. The center of the velocity circle determines the orientation of the orbit ($\vec{z}$

points in the direction of maximum speed) and its "shape" (Lz determines r_{max}/r_{min}).

3. From the velocity diagram, we can algorithmically determine the whole orbit.

III. Perturbations

9. *The Perturbation Formula; Radial Thrust*

It is in the study of perturbations, or how orbits change under small kicks other than those given by the sun, that our use of velocity diagrams really pays off. It pays off for a very good reason, which we mentioned in the introduction as a qualitative form of Newton's Second Law of Motion:

> Force acts on the paths of particles by changing *velocity* and *not position.*

If we fail to take account of this fact we may be faced with situations that appear counter-intuitive. For example, suppose a spaceship in a circular orbit around a planet applies a small outward thrust (Figure IIa).

How will the orbit change? "Intuition" may suggest that the orbit will elongate in the direction of the thrust, something like Figure IIb. In fact, the orbit will elongate, but in a direction *perpendicular* to the kick as in Figure IIc.

To understand this we consider how the kick changes the velocity diagram. The spaceship started in a circular orbit whose velocity diagram is centered on the origin. Since force affects velocity and not position, it is reasonable (and we shall show below) that the effect of the kick in velocity space *really is* just to move the velocity circle in the direction of the kick (Figure I2). The corresponding change in the position space orbit is the "counter-intuitive" effect described above.

Our strategy for studying perturbations will be to see how kicks change the

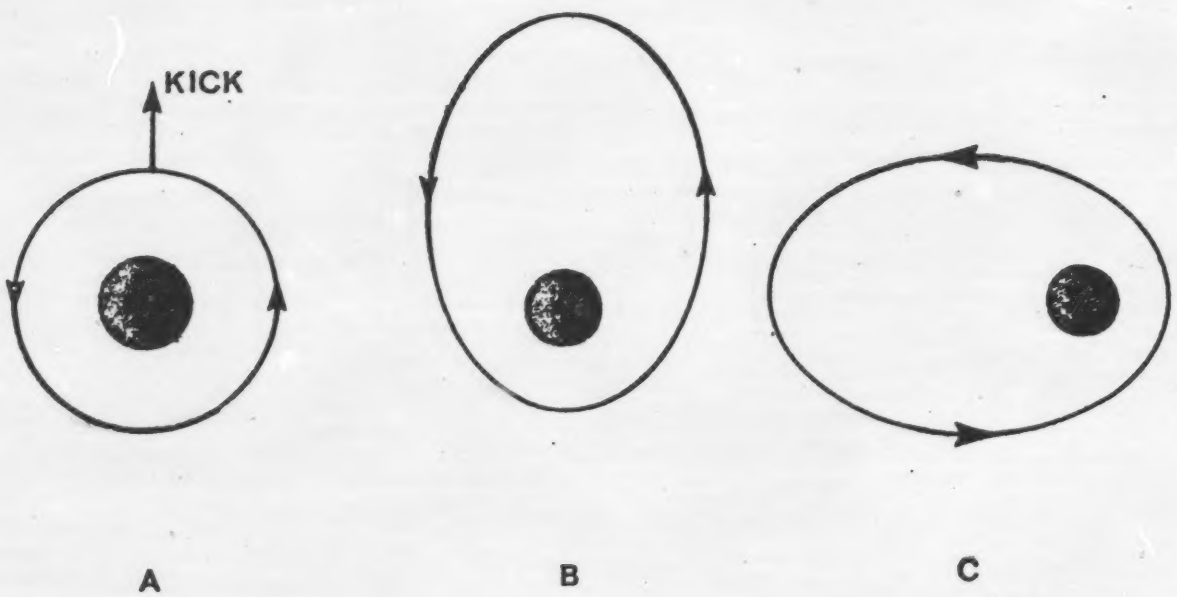


Figure 11: Will the outward kick on orbit (a) produce (b) or (c)?

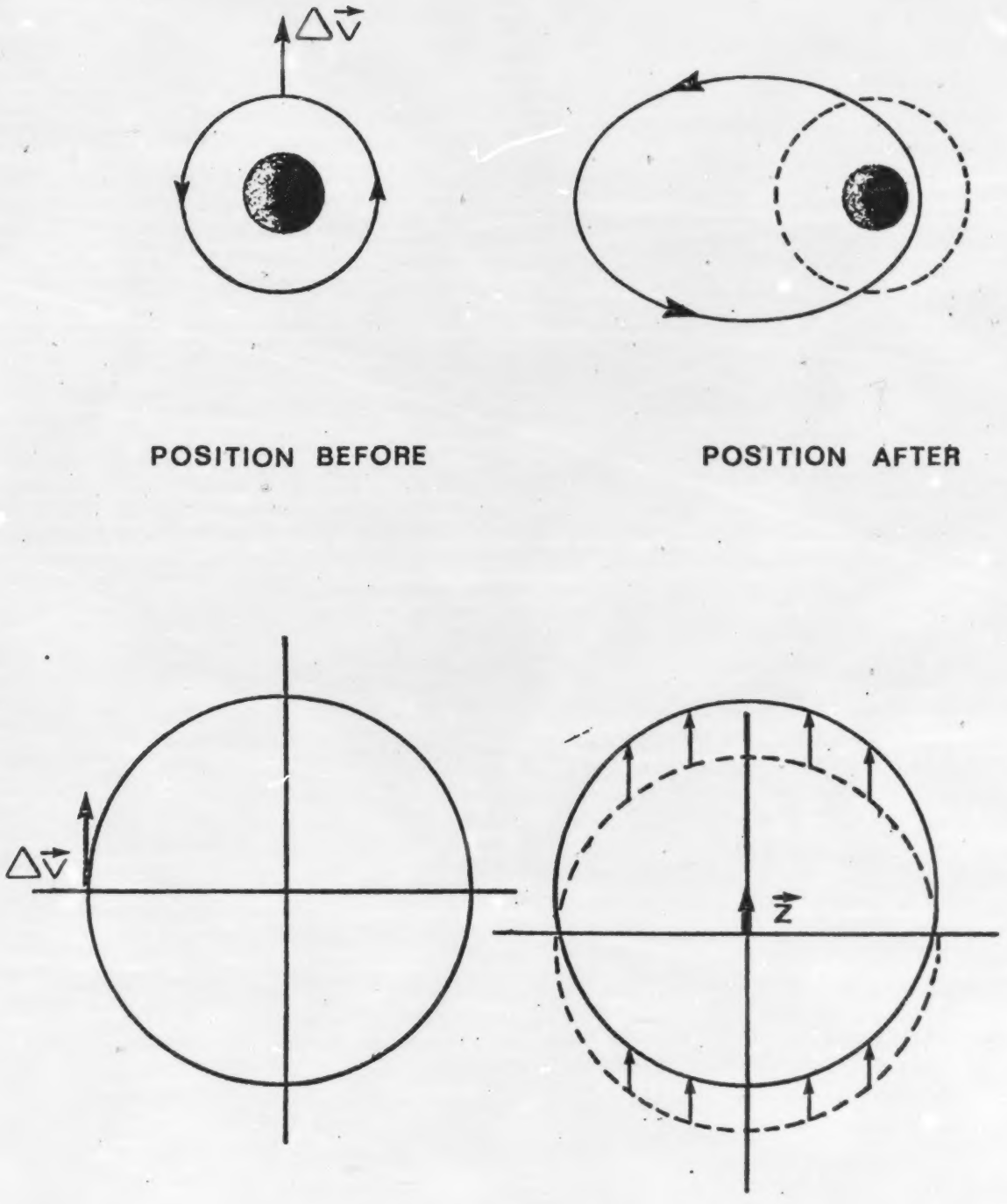


Figure 12: The perturbation induced by an outward kick.

velocity diagram. More precisely, we know that the shape and orientation of the orbit is determined by $L\vec{z}$, the $\vec{z}$ vector times the angular momentum, so we want to find the change in $L\vec{z}$, $\Delta(L\vec{z})$, produced by an arbitrary kick.

The basic velocity space equation, $\vec{v} = \vec{z} + \vec{u}$, gives $L\vec{z} = L\vec{v} - L\vec{u}$. Since $\vec{u}$ has length K/L we have $L\vec{u} = K\vec{s}$, where $\vec{s}$ is a vector of unit length whose direction is determined solely by the object's position. (It is perpendicular to the radial vector $\vec{r}$.) To compute the effect of a kick $\Delta\vec{v}$ on $L\vec{z} = L\vec{v} - K\vec{s}$, notice that since kicks do not affect position, $\vec{s}$ is unchanged. K is also unchanged. Therefore the changein $L\vec{z}$ is the same as the change in $L\vec{v}$, and the first-order approximation to the change in a product of changing quantities gives

Perturbation Formula: $\Delta(L\vec{z}) = \vec{v}\Delta L + L\Delta\vec{v}$

We can use the Perturbation Formula to tidy up our discussion of the "radial thrust problem" (Figure 14). Since $\Delta\vec{v}$ is in the radial direction, the angular momentum does not change ($\Delta L = 0$), so the formula implies $\Delta(L\vec{z}) = L\Delta\vec{v}$. This means that the velocity diagram changes from a $\vec{z}$ vector of zero to a $\vec{z}$ vector in the direction of $\Delta\vec{v}$ (Figure 12).

Intuitively, the velocity circle is "pushed" in the direction of the kick. Note that an inward kick at the bottom of the position space orbit would have the same effect.

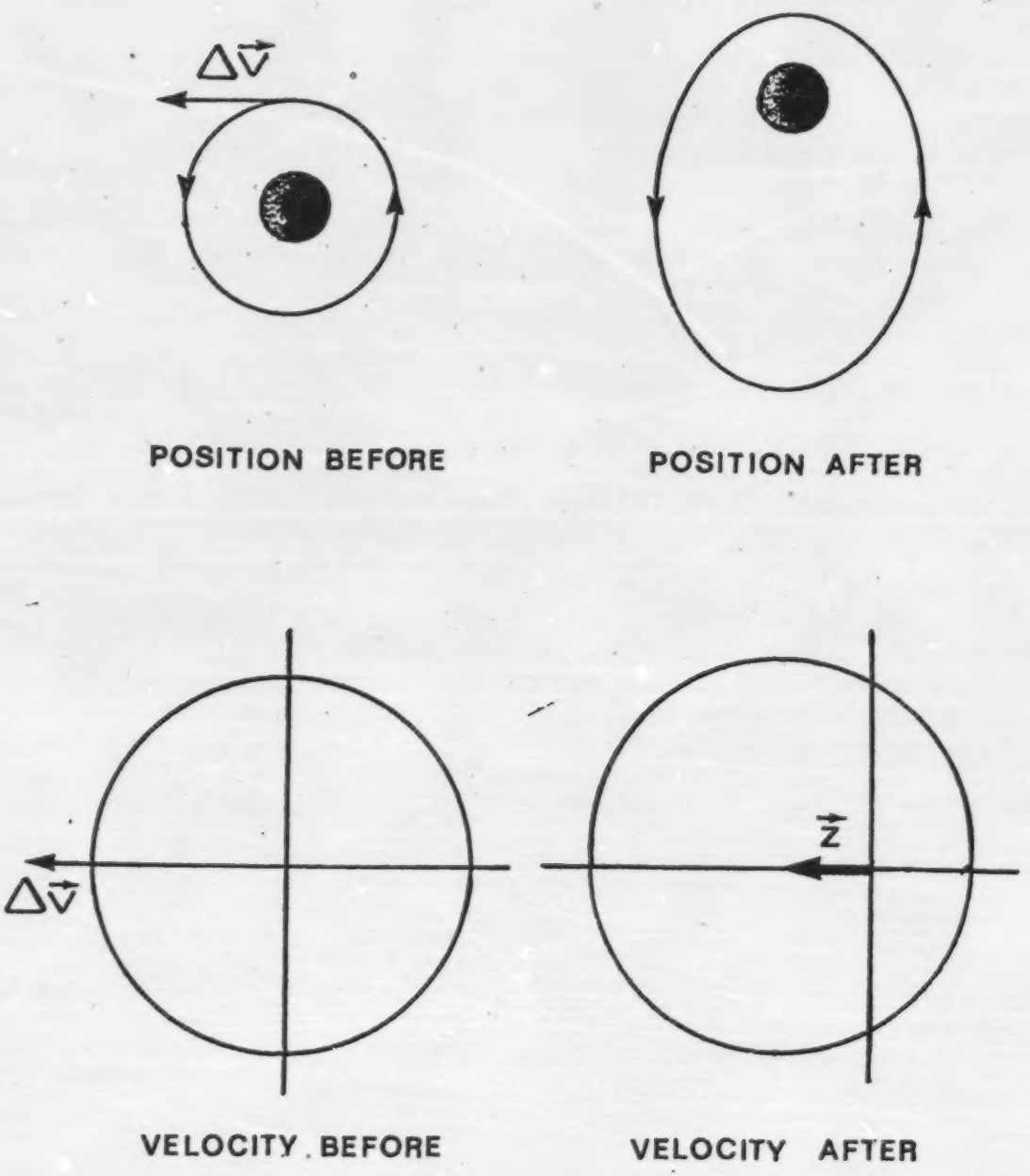


Figure 13: The perturbation induced by a tangential kick.

10. *Tangential Thrust; Solar Wind; $r^{-(2+\epsilon)}$ Laws*

In this section we apply the Perturbation Formula to some other orbit problems.

Tangential Thrust: Suppose again that a rocket starts in a circular orbit, but this time provides a tangential kick (Figure 13). To determine $\Delta(L\vec{z}) = \vec{v}\Delta L + L\Delta\vec{v}$ we note that $\vec{v}\Delta L$ is an impulse in the direction of $\Delta\vec{v}$ since ΔL is positive and $\vec{v}$ is parallel to $\Delta\vec{v}$. Then the newly created $\vec{z}$ vector must be in the direction of the impulse (Figure 13). The elongation in position space is again perpendicular to the kick.

The Solar Wind:[5] Assume the rocket is affected not only by the planet's gravity but also by a small constant force (Luehrmann's "solar wind"). If the perturbing force is small compared to gravity, each revolution of the rocket will be nearly an ellipse. We can therefore think of the orbit as an ellipse which varies through time. To compute how the ellipse changes we view the wind as providing impulses all along the orbit (Figure 14) and sum $\Delta(L\vec{z}) = \vec{v}\Delta L + L\Delta\vec{v}$ over one revolution.

The $L\Delta\vec{v}$ contribution is a net change in the direction of $\Delta\vec{v}$. To compute $\vec{v}\Delta L$ we notice that ΔL is positive on the bottom half of the orbit and negative on the top half as shown in Figure 15b. We can sum the $\vec{v}\Delta L$'s by exploiting the symmetry of the orbit. The vertical components of the $\vec{v}\Delta L$'s on the left cancel the vertical components of the $\vec{v}\Delta L$'s on the right, leaving only a horizontal component in the direction of $\Delta\vec{v}$ (Figure 15c). This adds with $L\Delta\vec{v}$ to produce a $\vec{z}$ vector in the direction of the wind. Intuitively, the velocity circle gets "blown" in the direction of the wind. The orbit elongates perpendicular to the wind as in Figure 16.

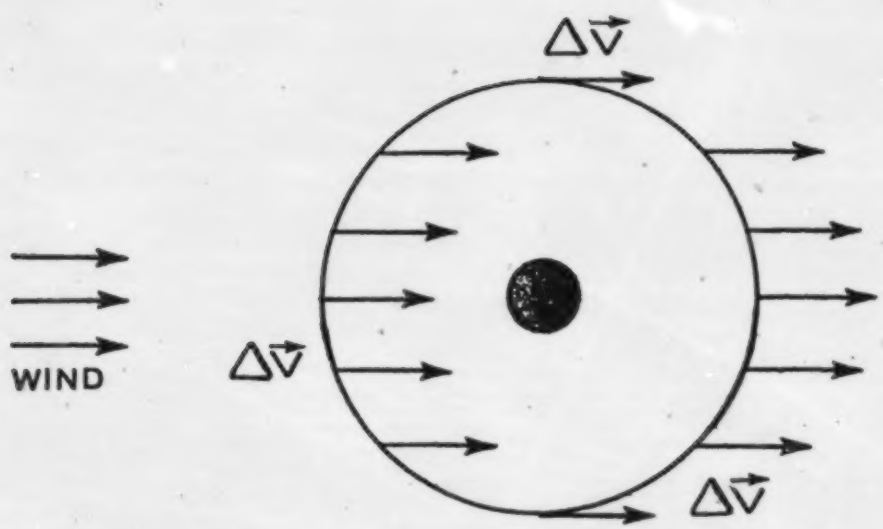


Figure 14: The solar wind.

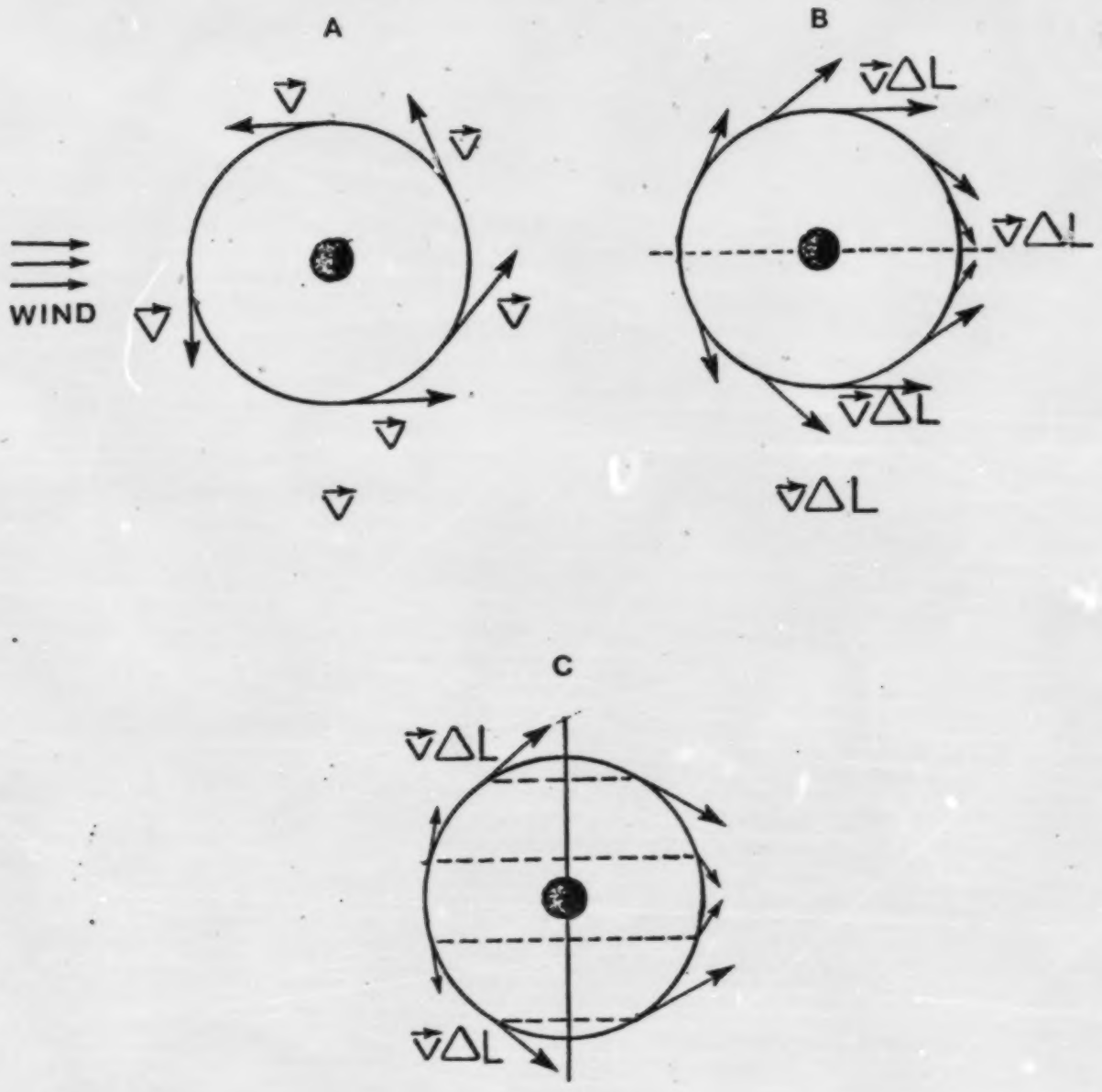


Figure 15: The vectors $\vec{v}$ (a) and $\vec{v}\Delta L$ (b) for the solar wind. In (c) we see that the vertical components of $\vec{v}\Delta L$ on the left cancel the vertical components on the right.

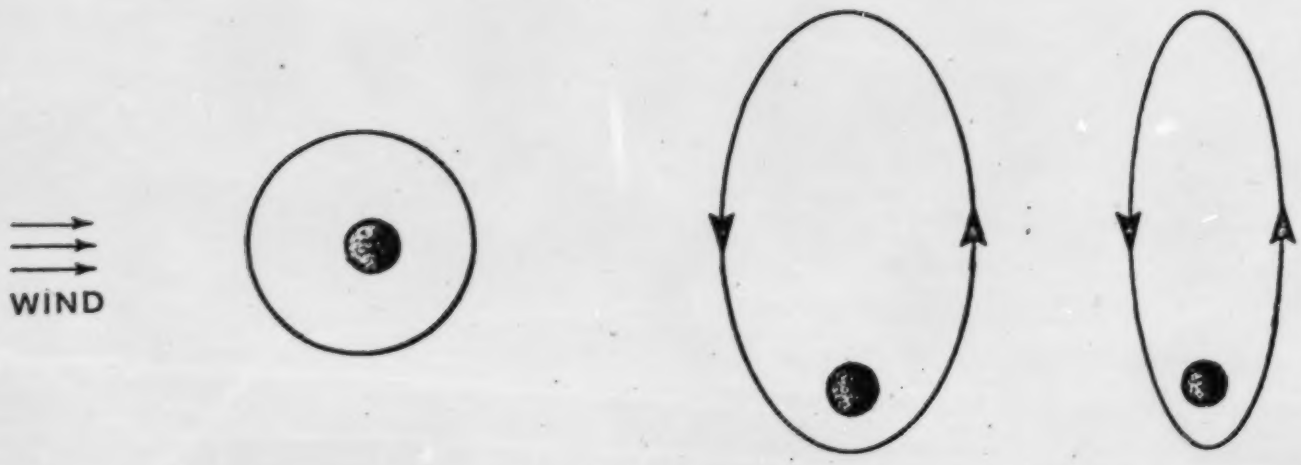


Figure 16: Position space change in the orbit under the solar wind.

Since the wind does not affect the symmetry of the orbit about the vertical axis we can apply the same analysis as above to show that the $\vec{Lz}$ vector continues increasing in the direction of the wind. From this we conclude that the orbit becomes more and more eccentric while the direction of $\vec{z}$ remains constant, and the orbit continues to elongate *perpendicular* to the wind.

The orbit becomes closer and closer to a straight line, and it eventually reaches a point where the "small" wind can have large qualitative effects over a timescale of less than one revolution (the orbit in fact reverses direction), and our method of averaging over an entire revolution becomes inappropriate.

The $r^{-(2+\epsilon)}$ Force Field: If ϵ is a small constant (we will take it to be positive), we can treat the central force field of magnitude $r^{-(2+\epsilon)}$ as a perturbation of the r^{-2} field. The perturbing force is some force (positive or negative) in the radial direction. To understand how this perturbation affects the orbit, we make the important observation that the shape of a r^{-2} or $r^{-(2+\epsilon)}$ orbit does not depend on the scale which we use to measure radius. Therefore we can determine shape by using *any* scale which makes it convenient to compute the effect of the perturbation. For the orbit shown in Figure 17, we scale to make the distance OP equal to one. Since $1/r^2 < 1/r^{(2+\epsilon)}$ for $r < 1$ and $1/r^2 > 1/r^{(2+\epsilon)}$ for $r > 1$ the perturbing force is as shown.

For this perturbation the kicks are radial, so L is constant. This means $\Delta(L\vec{z}) = L\Delta\vec{z}$, but from the perturbation formula, $\Delta L = 0$ implies $\Delta(L\vec{z}) = L\Delta\vec{v}$. Hence $\Delta\vec{z} = \Delta\vec{v}$. Now we can sum $\Delta\vec{z} = \Delta\vec{v}$ over an entire orbit. The left-right symmetry of the orbit and perturbing force means that the sum of horizontal components of the kicks must cancel,

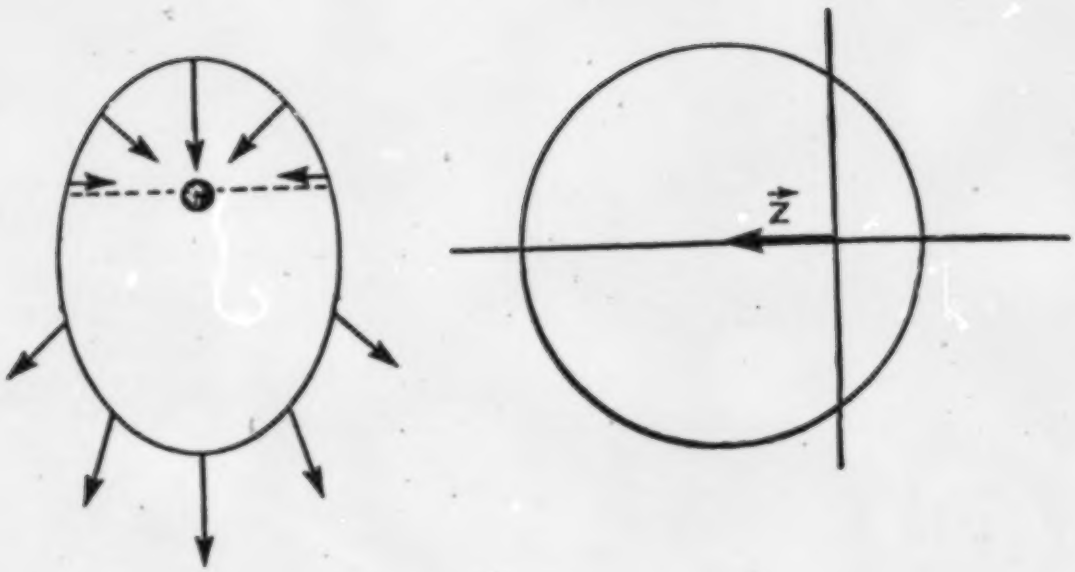


Figure 17: Orbit with perturbing force indicated for $r^{-(2+\epsilon)}$.

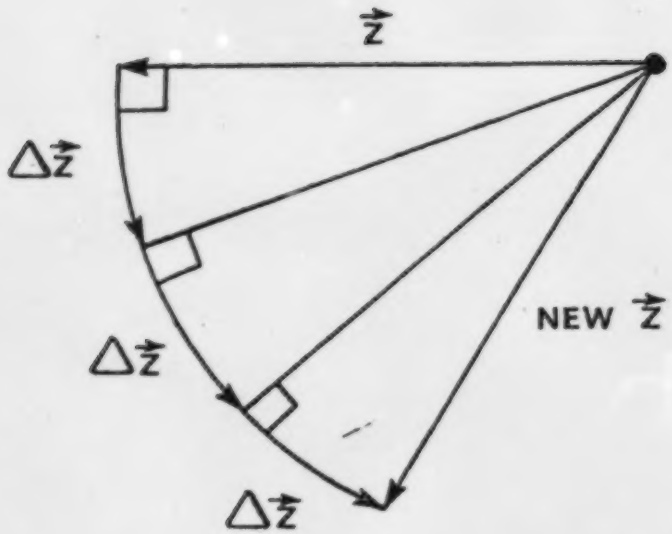


Figure 18: $\Delta\vec{z}$ perpendicular to $\vec{z}$.

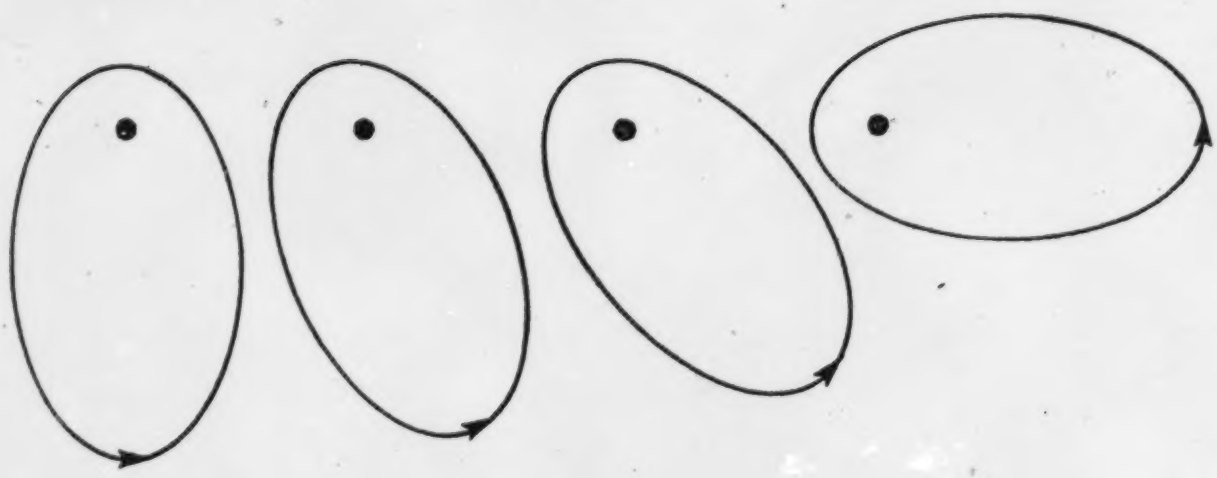


Figure 19: Precession of $r^{-(2+\epsilon)}$ orbit.

and the net $\Delta\vec{z}$ is downward, that is, perpendicular to the original $\vec{z}$. Subsequent $\Delta\vec{z}$'s will be perpendicular to the current $\vec{z}$ and this results primarily in a rotation of $\vec{z}$, not a change in length (Figure 18). The "major axis" of the position space orbit, by the consequences we derived from Correlation of Angle, must follow this counter-clockwise rotation. Though the orbit retains its shape $L\vec{z}$, it precesses (Figure 19).

Warning: It should be remarked that in the preceding two examples we looked at the $\vec{F}$ vectors as representing $\Delta\vec{v}$ for the perturbation formula. Of course we should have used $\vec{F}\Delta t$, but, because of the symmetry involved, the Δt factor can be ignored in those two cases. In more complicated situations, though, this does become an issue. For example, we invite the reader to use the techniques of this section to treat the perturbation induced by an oblate sun.

IV. Analytic Results

II. *The Orbit is a Conic Section*

An objection that is sure to occur to some of our readers goes something like this: "All these intuitive methods are fine, but if you want useful quantitative information you have to return to the standard differential equations you've been trying to get along without."

Of course there are orbit problems our simple methods won't handle. As for the standard results, however, we are able to derive the orbital equation directly from our velocity diagram using no more than trigonometry:

The orbit is described in polar coordinates by the equation

$$r = \frac{L}{u - z\cos\theta}$$

The proof is a natural correlation of the basic quantities, $\vec{r}$, $\vec{v}$, $\vec{u}$, $\vec{z}$ and L using the definition of angular momentum. At a point in the orbit when the $\vec{u}$ vector and the $\vec{z}$ vector differ in direction by an angle θ we construct the angular momentum triangle (see appendix).

The area of the triangle OAB in Figure 20 is by definition L/2. If h is the height of the triangle then

$$\frac{1}{2}L = \frac{1}{2}rh$$

Since u and r are perpendicular, the height of the triangle is given by $h = u - z\cos\theta$. Therefore

$$L = rh = r(u - z\cos\theta).$$

Here θ represents the angle in *velocity space* between $\vec{u}$ and the fixed vector $\vec{z}$ Correlation of Angles implies that θ also measures the angle in position space from $\vec{r}$ to a fixed vector

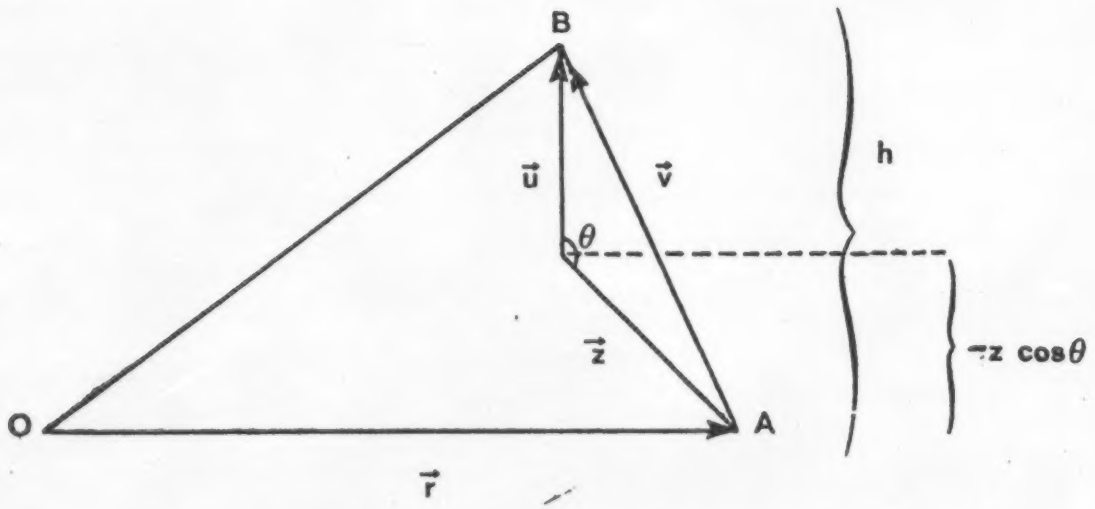


Figure 20: The angular momentum triangle OAB.

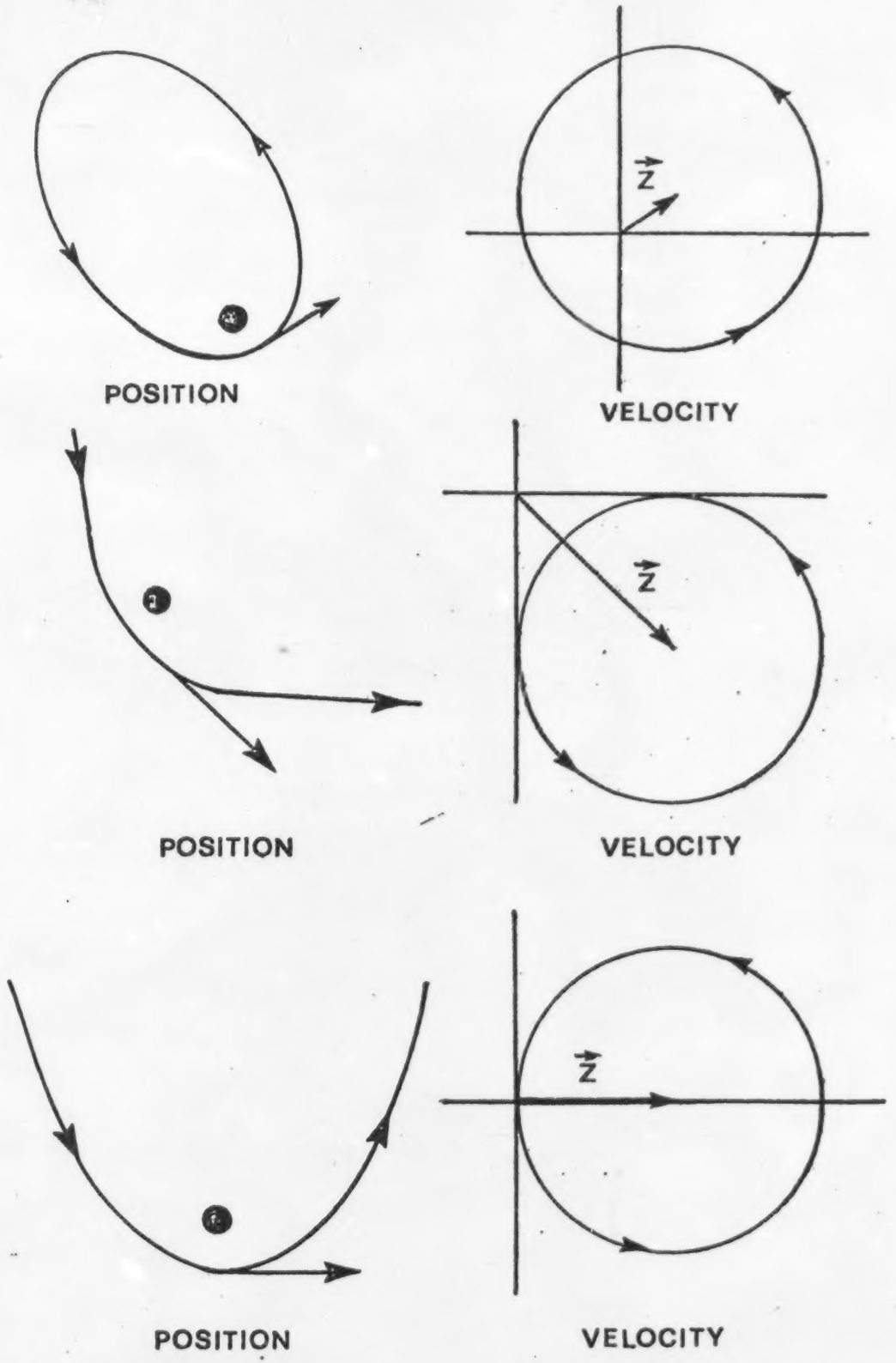


Figure 21: Sample orbits in position and velocity space.

perpendicular to $\vec{z}$. Therefore r and θ are polar coordinates in position space.

The above equation describes a conic section. When the origin of velocity space lies within the circle, u>z and the orbit is an ellipse. When the origin is outside the circle, u < z and the orbit is hyperbolic. When the circle passes through the origin, u = z and the orbit is parabolic (Figure 21).

Writing the equation in the form

$$r = \frac{L/u}{1-\left(z/u\right)\cos\theta} = \frac{L^2/K}{1-\left(Lz/K\right)\cos\theta}$$

we get the "standard form" for a conic section and see that Lz/K is the eccentricity, and L^2/K is the radius of the orbit when the eccectricity is zero.

12. *Conservation of Energy*

Energy conservation does not arise naturally using this geometric approach although we can obtain the result as a simple application.

Apply the law of cosines to the velocity diagram in Figure 22 to get

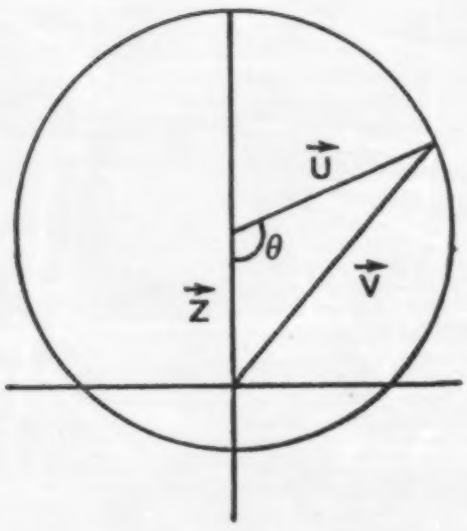


Figure 22: $v^2 = z^2 + u^2 - 2uz \cos\theta$.

$$v^2 = z^2 + u^2 - 2uz\cos\theta$$

Substituting $z\cos\theta = u - \frac{L}{r}$ (from Section II) we obtain

$$v^2 = z^2 - u^2 + \frac{2uL}{r}$$

and hence (since $u = K/L$)

$$\frac{v^2}{2} - \frac{K}{r} = \frac{z^2 - u^2}{2}$$

Since z and u are constant for the orbit, $\frac{z^2 - u^2}{2}$ is a constant, the total energy, E. It is interesting to note that when the planet crosses the semi-minor axis ($\vec{r}$ perpendicular to $\vec{v}$) $\vec{u}$, $\vec{z}$, and $\vec{v}$ form a right triangle with $v^2 = u^2 + z^2$; hence the kinetic energy $v^2/2$ is exactly the negative of the total energy there.

13. *Kepler's Third Law*

We can use the relation of angular momentum to area swept out, $2A = Lt$, to compute the period of the planet's revolution. In one period the planet sweeps out the entire area of its elliptical orbit. The area of an ellipse of semi-major axis a and eccentricity e is given by $A = \pi a^2\sqrt{1-e^2}$. For the orbit we have

$$a = \frac{1}{2}\left(r_{min} + r_{max}\right) = \frac{1}{2}\left(\frac{L}{u-z} + \frac{L}{u-z}\right)$$

$$= \frac{Lu}{u^2-z^2} = \frac{K}{u^2-z^2}$$

The eccentricity is e = z/u, so $\sqrt{1-e^2} = \frac{\sqrt{u^2-z^2}}{u} = \frac{\sqrt{K}}{u\sqrt{a}}$

Then the period is determined by

$$LT = 2\pi a^2\sqrt{1-e^2} = \frac{2\pi a^{3/2}\sqrt{K}}{u\sqrt{a}}$$

or

$$T = \frac{2\pi a^{3/2}\sqrt{k}}{Lu} = \frac{2\pi}{\sqrt{k}}a^{3/2}$$

In terms of quantities appearing in the velocity diagram we get

$$T = \frac{2\pi K}{(u^2-z^2)^{3/2}} = \frac{2\pi K}{(-2E)^{3/2}}$$

14. *Open Orbits*

Rather than treat the hyperbolic case in detail, we leave the reader to verify the following:

1. For an open orbit, the arc of the velocity circle which is actually traversed is the part shown below, bounded by the tangents to the circle through the origin of velocity space. Velocity space geometry gives the correlation of energy with limiting velocity, v

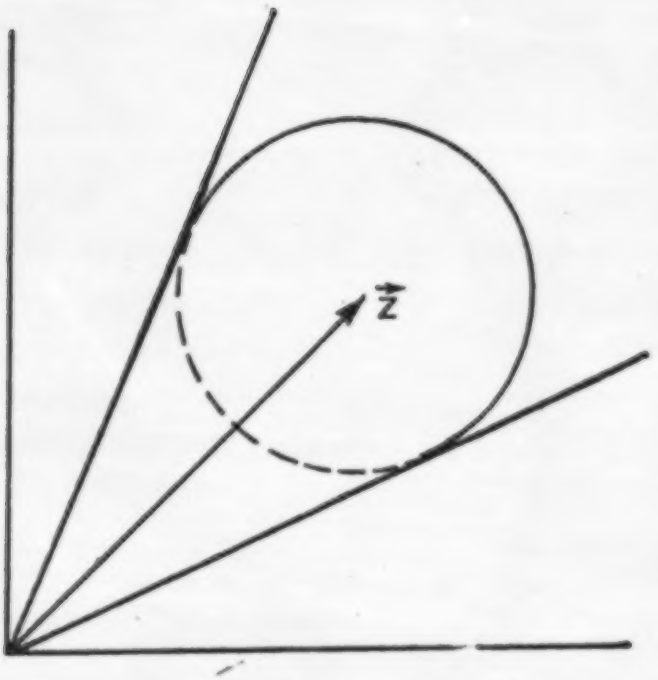


Figure 23: The velocity diagram for an open orbit.

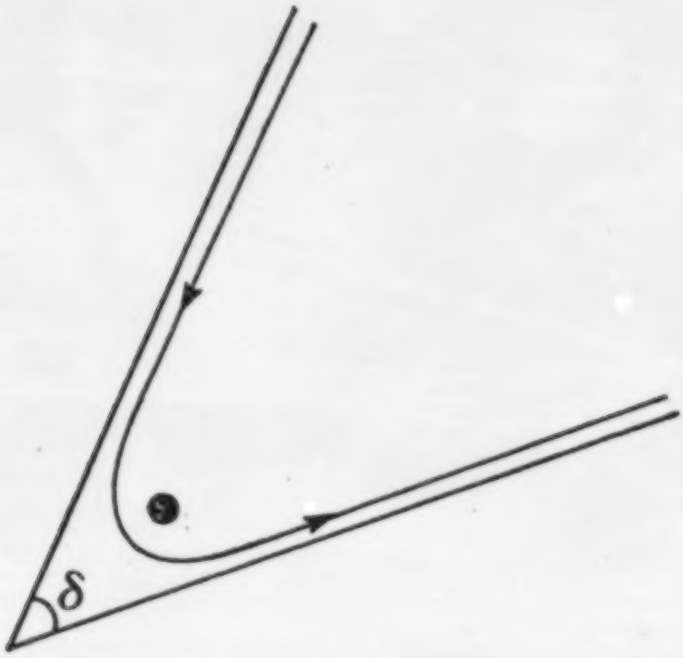


Figure 24: Deflection angle for an open orbit.

$-\sqrt{z^2-u^2}$ $-\sqrt{2E}$ (Figure 23).

2. The deflection angle (angle between the two asymptotes of the hyperbola) can be easily found as a function of energy and angular momentum: $\tan\frac{\delta}{2} = \frac{\sqrt{2E}}{K}$ (Figure 24)

15. *Suggestions for Further Research*

We have by no means exhausted the study of the geometry of orbits in this paper. The geometry of orbits, particularly the perturbation theory, is a rich source of problems, even of mini-research projects of the type described by Luehrmann[5]. Below we make some suggestions for problems and study topics.

An instructive paradox: The Galilean transformation requires the relation between velocities measured in two different frames of refrence moving with relative velocity $\vec{v}_0$ to be $\vec{v}' = \vec{v} + \vec{v}_0$. This it is a simple matter to move into a frame with relative velocity $-\vec{z}$ and transform the $\vec{z}$ vector for an elliptical orbit to 0. Why then does one not observe a circular orbit in the new frame? In particular, what fails in the algorithm of Section 8 which *does* generate a circle in position space given a circle centered about the origin in velocity space?

An Aid to Astrogation? Suppose we had to pilot a spaceship in a gravitational field (such as simulated in computer "space war" games). Would a velocity diagram be a useful addition to our instrument panel? For example, to change from an elliptical orbit to a circular orbit

we need only consult the velocity diagram and apply a force to cancel the $\vec{z}$ vector. On the other hand, lots of information is lacking if we use *only* the velocity picture. Intercepting another spaceship is a tricky problem involving timing. (Although merely matching its orbit is easy.) What other instruments should supplement or possibly replace a velocity diagram?

Geometry of the harmonic oscillator: A fundamental geometric property of solutions to the $1/r^2$ differential equation is that they have a vector constant of motion ($\vec{r}$, or the maximum velocity, or the Runge-Lenz vector are all possible choices for this constant). The two dimensional harmonic oscillator with equal mode frequencies has a similar structure. Solutions have an obvious axis which may also be assigned a magnitude in any number of ways. Can one develop a useful velocity space geometry for that system? Can one treat simple perturbations, as with orbits?

More Solar Wind: A further discussion of the solar wind phenomenon could make use of the fact that the force field is conservative and therefore the energy of the orbit is constant. This implies that the angular momentum decreases as the orbit becomes more eccentric. Thus the velocity circle is not only "blown by the wind" but also the radius becomes larger and finally infinite when the orbit degenerates to a line. Show that the changing velocity circle always passes through two fixed points in the plane. (Figure 25)

Our method of averaging over entire orbits is only a first-order perturbation theory whereas the formula

$$\frac{d(L\vec{z})}{ds} \qquad L\frac{d\vec{v}}{ds} + \vec{v}\frac{dL}{ds}$$

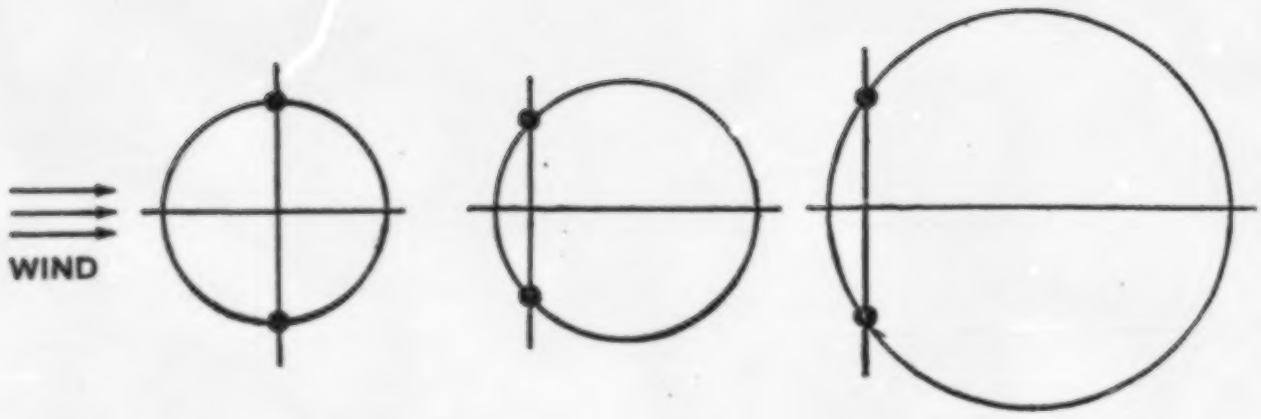


Figure 25: Changing velocity circle under solar wind.

(s is *any* parameter) is exact. One would like to have a treatment of the solar wind which works near the turnaround points, z/u approaching 1.

Finally, is there a complete perturbation theory based on the geometry of orbits? In particular how can one treat perturbations out of the plane of the orbit?

Appendix: *Angular Momentum and Kepler's Second Law*

Throughout this paper we have been assuming Kepler's Second Law. There is a simple geometric proof of this which we place here in an appendix because it did not originate with us. It can be found in Newton's Principia!

The angular momentum, which we denote by L, is defined to be twice the area of the triangle determined by the velocity vector and the radius vector from the sun to the planet. As shown below L is constant if the velocity doesn't change -- the triangles have equal areas since they have equal bases (the length of $\vec{v}$) and equal heights (Figure A1). More remarkably, *L remains constant if we change the velocity by applying any kick in the radial direction (towards or away from the sun).* The effect of a kick $\Delta\vec{v}$ on the angular momentum triangle is illustrated below (Figure A2). The kick changes $\vec{v}$ to $\vec{v}'$, but triangles OPQ and OPQ' have the same base, OP, and the same height (h in the diagram) since OP and QQ' are parallel. Therefore OPQ and OPQ' have the same area, and angular momentum is unchanged.

A planet moving about the sun, subject to no force but the sun's gravitation, has every kick applied in the radial direction. None of these change the angular momentum, which is therefore an invariant of the planet's orbit. To find a geometric interpretation of this fact, we examine the orbit at time intervals Δt small enough that the velocity does not change much over each interval. In each interval the radius vector sweeps out a small triangle. The area of one of these small triangles is

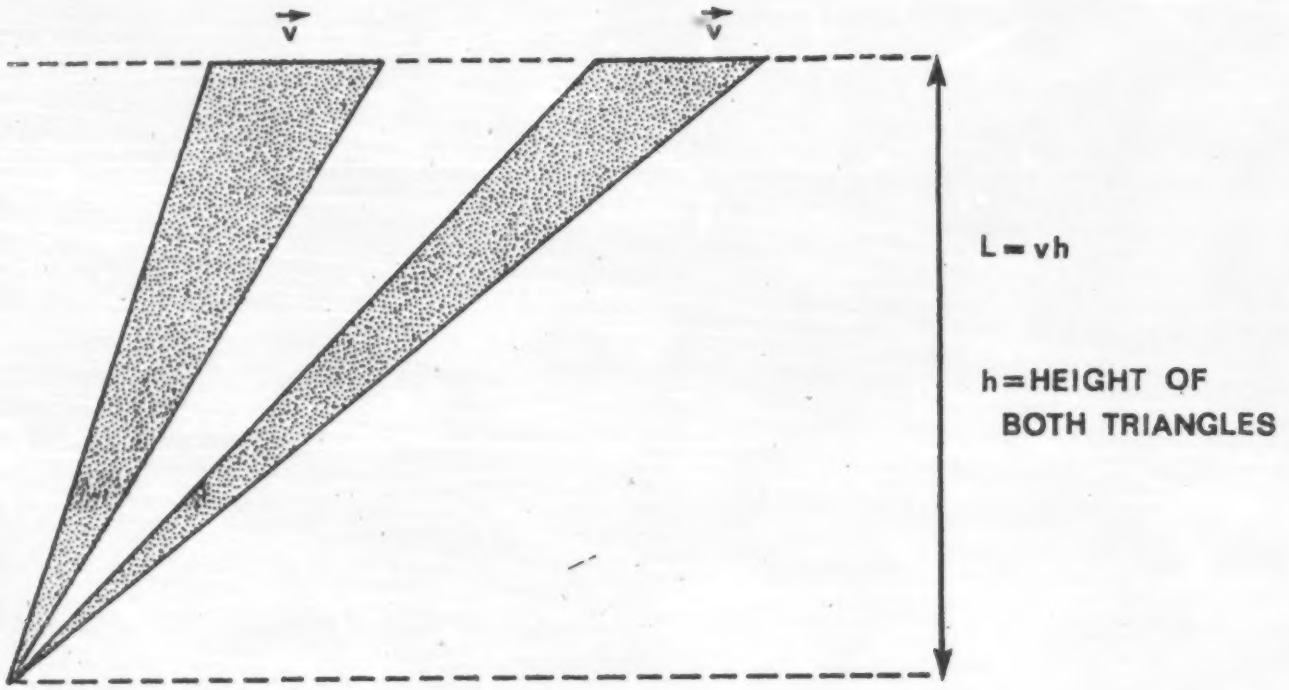


Figure A1: Triangles of equal area

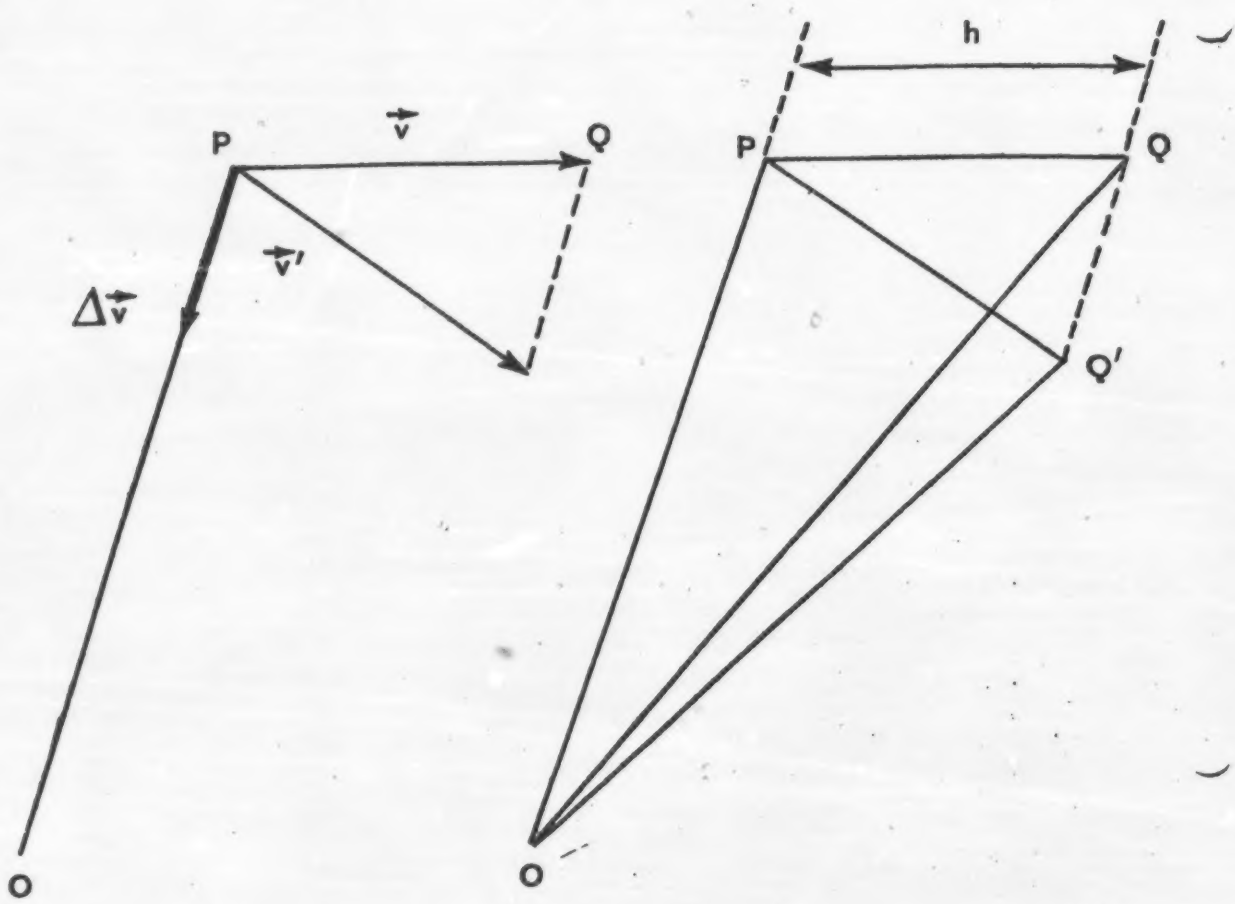


Figure A2: Area OPQ = Area OPQ'

$$\frac{\text{(base) x (height)}}{2} = \frac{1}{2} v \Delta t \times h = \frac{1}{2} L \Delta t$$

(h as in Figure A1). The total area swept out over some long time

$$T = \Delta t_1 + \Delta t_2 + \ldots$$

is the sum of the areas of the small triangles

$$\begin{aligned} A &= \frac{1}{2} L \Delta t_1 + \frac{1}{2} L \Delta t_2 + \ldots \\ &= \frac{1}{2} L (\Delta t_1 + \Delta t_2 + \cdots) \\ &= \frac{1}{2} L T \end{aligned}$$

This gives Kepler's Second Law:

> For a body moving in a radial force field, the radial vector sweeps out equal areas in equal times.

It is unfortunate that this proof is not more often presented in physics courses (although Feynman [6] discusses it and there is a movie [7] demonstrating this argument).

Notes and References

1. S. Papert, "Uses of Technology to Enhance Education". AI Memo #298, MIT Artificial Intelligence Laboratory, Cambridge, Mass., 1973.

2. We consistently set m = 1, as the mass parameter is quite irrelevant to these orbital discussions.

3. This fact is one of the first theorems in Papert's Turtle Geometry. In that context it is an empirical theorem routinely used by children in directing a computer- controlled "turtle" to draw circles. See S. Papert, "Teaching Children to be Mathematicians vs. Teaching Them about Mathematics", *Int. J. Math. Educ. Sci. Technol.*, Vol 3, 249-262, 1972.

4. This is another theorem in Turtle Geometry. It is called the Total Turtle Trip Theorem, and states, "If a turtle moves around a simple closed curve its heading changes by precisely 360°." Here, the heading corresponds to the direction of the velocity. See Reference 1.

5. This phenomenon was introduced to us by Arthur Luehrmann whose article "Orbits in the Solar Wind", *Am. J. of Phys.*, Vol. 42/5, May 1974, indicates many areas of research dealing with the solar wind.

6. R. P. Feynman, *The Character of Physical Law*, Cambridge, MIT Press, 1965.

7. B. Cornwall, C. Cornwall, and A.M. Bork, *Newton's Equal Areas*, International Film Bureau, Chicago, 1967.